REMARKS

This Amendment is being filed in response to the Office Action mailed from the U.S. Patent and Trademark Office on October 29, 2004, in which claims 1-86 were rejected. With this Amendment, claims, 1, 14, 19, 33, 38, 52, 57, 71 and 78 are amended, and claims 23, 42 and 61 are cancelled. As such, Applicants respectfully request reconsideration and allowance of pending claims 1-22, 24-41, 43-60 and 62-86.

The Office Action rejected claims 14, 33, 52 and 71 under 35 U.S.C. 101 as lacking utility. The Office Action rejected claims 1-86 under 35 U.S.C. 103 as being unpatentable over U.S. Patent No. 5,391,144 to Sakurai et al. ("the Sakurai et al. '144 patent"). Also, the Office Action rejected claims 1-86 under the judicially created doctrine of obviousness-type double patenting as being unpatentable over various combinations of Applicants' patents and copending patent applications.

The Office Action rejected claims 14, 33, 52 and 71 under 35 U.S.C. 101 as lacking utility. The Office Action states at page 2:

> Claims 14, 33, 52 and 71 are rejected under 35 U.S.C. 101 because the disclosed invention is inoperative and therefore lacks utility. These limitations state that the light transmitting element is used to obtain optical data and that these optical data is an MR image. This is inoperable in that magnetic resonance has nothing to do with light transmitting elements and hence this is inoperable.

With this Amendment, Applicants have amended claims 14, 33, 52 and 71 to delete the term "wherein the optical data is a magnetic resonance image" and replace it with "further comprising an imaging system selected from the group consisting of ultrasound, magnetic resonance imaging, endoscope and laproscope". Therefore, this rejection is overcome. As such, Applicant respectfully requests reconsideration and allowance of claims 14, 33, 52 and 71.

Obviousness Rejections Under 35 U.S.C. 103

The Office Action rejected claims 1-86 under 35 U.S.C. 103 as being unpatentable over Sakurai et al. '144. The Office Action states on pages 4-5:

> **Sakurai et al.'144 teach** all the elements of the current invention including a device for removing occlusions in blood vessels (col. 1, lines 12-53) comprising: an ultrasonic probe having a proximal end and a distal end (see in Figure 13, alternative ultrasonic probes 61 and 62, both having a proximal end and a distal end); a sound conductor with a proximal end and a distal end, said distal end being connected to the coupling assembly and said proximal end being connected to a transducer capable providing ultrasound energy (see Figure 13, element 63; and also see col. 12, lines 45-68 and col. 13, lines 1-6), to transmit ultrasound energy from said transducer to said probe, **causing said probe to be oscillated in a substantially transverse mode to the probe longitudinal axis and wherein the probe is capable of supporting standing transverse sound waves** to cause generation of ultrasonic cavitation energy in at least one location along the longitudinal axis of the ultrasonic probe (in Figure 13, see how the horn (element 63) transmits the vibrations to the ultrasonic probes (elements 61 or 62), through the attachment means (elements 73a and 73b); **regarding the transverse oscillation of the probe see Figures 40 and 41 and indications of anti-node or loops wherein there is maximum oscillation along the length of the probes)** and wherein the sound conductor and transducer (vibrating at an ultrasonic frequency) are in the device handle as illustrated by Figure 13.
>
> ...
>
> Sakurai et al. '144 do not teach use of the probe for destruction of tumor cells in the uterus. It would have been obvious to one skilled in the art at the time that the invention was made to have utilized the apparatus and procedure at any desired area of interest for destruction and removal of the tissue of interest. Therefore, it would have been obvious to one skilled in the art at the time that the invention was made to have emulsified and/or fragmented the tissue of interest such as destruction of cancer cells in the uterus.

"Obviousness can only be established by combining or modifying the teachings of the prior art to produce the claimed invention where there is some teaching, suggestion, or motivation to do so found either explicitly or implicitly in the references themselves or in the knowledge generally available to one of ordinary skill in the art." M.P.E.P. 2143.01. "The test

for an implicit showing is what the combined teachings, knowledge of one of ordinary skill in the art, and the nature of the problem to be solved as a whole would have suggested to those of ordinary skill in the art." *In re Kotzab,* 217 F.3d 1365, 1370, 55 U.S.P.Q.2d 1313, 1317 (Fed. Cir. 2000). See also *In re Fine*, 837 F.2d 1071, 5 U.S.P.Q.2d 1596 (Fed. Cir. 1988); *In re Lee*, 277 F.3d 1338, 1342-44, 61 U.S.P.Q.2d 1430, 1433-44 (Fed. Cir. 2002); *In re Jones*, 958 F.2d 347, 21 U.S.P.Q.2d 1941 (Fed. Cir. 1992); M.P.E.P. 2143.01.

With this Amendment, Applicants have amended independent claims 1, 19, 38, 57 and 78 to recite a device **wherein the ultrasonic energy creates a standing transverse wave in the probe producing a plurality of nodes and a plurality of anti-nodes.** Dependent claims 14, 33, 52 and 71 are also amended. No new matter is added with this Amendment. Support for these amendments to the claims is found throughout Applicants' specification and filed, and in at least the following passages:

> The ultrasonic medical device includes an elongated probe having a proximal end, a distal end, and a longitudinal axis wherein the elongated probe can be inserted into a body lumen, and an ultrasonic generator for providing ultrasonic energy to the elongated probe for emission along the longitudinal axis of the elongated probe to the tissue adjacent the body lumen, wherein **the ultrasonic energy creates a standing transverse wave in the elongated probe such that a plurality of nodes and a plurality of anti-nodes are formed along the longitudinal axis of the elongated probe to treat the tissue adjacent the body lumen.** (Applicants' specification; Page 9, Lines 7-16) (Emphasis Added).

> "**Probe**" as used herein refers to a device capable of being adapted to an ultrasonic generator means, which **is capable of propagating the energy emitted by the ultrasonic generator means along its length, resolving this energy into effective cavitational energy at a specific resonance (defined by a plurality of nodes and a plurality of anti-nodes at pre-determined locations along an "active area" of the probe**) and is capable of acoustic impedance transformation of ultrasound energy to mechanical energy. (Applicants' specification; Page 16, Lines 11-16) (Emphasis Added).

> **Transverse vibration** of the probe of an ultrasonic medical device **generates a plurality of nodes and a plurality of anti-nodes of cavitation energy along the longitudinal axis of the probe,** which are resolved into a plurality of cavitational anti-nodes emanating radially from the nodes and anti-nodes at specific points along the active portion of the

> probe. (Applicants' specification; Page 17, Lines 11-15) (Emphasis Added).
>
> By eliminating the axial motion of the probe and **allowing transverse vibrations only**, the active probe can **cause fragmentation** of large areas **of tissue spanning the entire length of the active portion of the probe due to generation of multiple cavitational anti-nodes along the probe length perpendicular to the probe axis**. (Applicants' specification; Page 20, Lines 3-6) (Emphasis Added).
>
> **Transverse oscillation of the probe 20 generates a plurality of cavitation anti-nodes 42 along the longitudinal axis of the probe 20, thereby efficiently destroying the tissues that come into proximity with the energetic anti-nodes 42**. (Applicants' specification; Page 22, Lines 16-18) (Emphasis Added).
>
> In another embodiment of the present invention, **the ultrasonic medical device is used with an imaging system, such as ultrasound, magnetic resonance imaging, an endoscope, or a laproscope**. (Applicants' specification; Page 12, Lines 3-5) (Emphasis Added).

The present invention relates to an medical device having a standing transverse wave producing a plurality of nodes and a plurality of anti-nodes. The cited prior art reference of the Sakurai et al. '144 patent <u>does NOT</u> disclose or suggest a medical device having a standing transverse wave.

Regarding the Sakurai et al. '144 patent, Applicants respectfully disagree with the Office Action's statement on p. 5 that the Sakurai et al. '144 patent teaches a "probe to be oscillated in a substantially transverse mode to the probe longitudinal axis and wherein the probe is capable of supporting standing transverse sound waves supporting a transverse ultrasonic vibration." The Sakurai et al. '144 patent does not disclose standing transverse waves or transverse ultrasonic vibration. In fact, the Sakurai et al. '144 patent does NOT contain any form of the word "transverse." As shown in the detailed analysis below, the Sakurai et al. '144 patent discloses longitudinal ultrasonic vibration, and does NOT disclose standing transverse waves or transverse ultrasonic vibration.

Examination of the Sakurai et al. '144 patent shows the Sakurai et al. '144 patent discloses longitudinal ultrasonic vibration and not standing transverse waves or transverse ultrasonic vibration. Differentiation between the Sakurai et al. longitudinal vibration and the

Applicants' claimed transverse vibration is evidenced by well known wavelength and stress equations, with further confirmation from the specification of the Sakurai et al. '144 patent.

Sakurai et al. Discloses Longitudinal Vibration – Wavelength Equation Confirmation

Sakurai et al. '144 patent discloses an ultrasonic treatment apparatus that uses longitudinal vibration. Sakurai et al. discloses:

> The ultrasonic oscillator 2 can vibrate in various modes which are represented in the graph, i.e., the lower half of FIG. 40. As may be understood from the graph, the oscillator 2 has indefinite number of vibration modes. **Among these vibration modes are: the fundamental mode; the second harmonic mode in which ultrasonic waves have half the length of those in the fundamental mode; the third harmonic mode in which ultrasonic waves have a third of the length of those in the fundamental mode; and the fourth harmonic mode in which ultrasonic waves have a quarter of the length of those in the fundamental mode.** (Sakurai et al. '144 patent; Col. 23, Line 64 - Col. 24, Line 7)(emphasis added).

The above statement in Sakurai et al. conforms with the equation for the wavelength of **longitudinal vibration** waves produced in a rod:[1]

$\lambda_n = \frac{2\pi c}{\omega_n}$ where c is the speed of sound and ω_n is the angular frequency

Substituting the expression for the angular frequency ω_n in terms of the linear frequency, ν_n,

$$\omega_n = 2\pi\nu_n$$

gives

$$\lambda_n = \frac{c}{\nu_n}$$

and finally substituting the expression for normal mode frequencies,

$$\nu_n = \frac{nc}{2L}$$

[1] Morse, P.M. and Ingard, K.U., *Theoretical Acoustics*, Princeton University Press, Princeton, NJ, pp. 116-120, 1968 (see attached Exhibit A).

yields

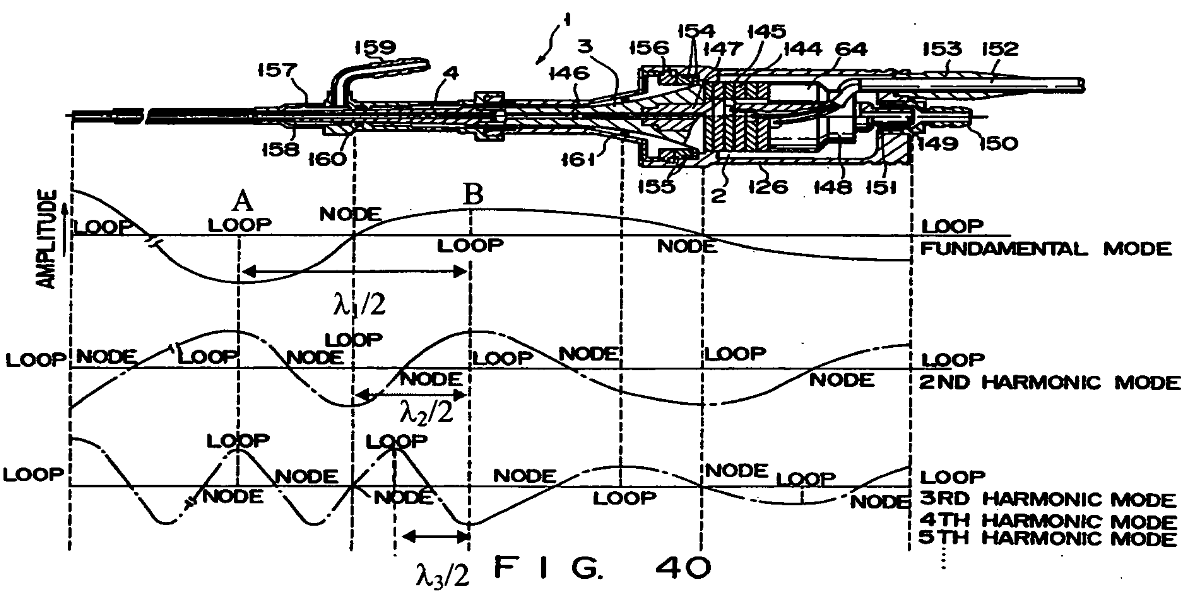
$$\lambda_n = \frac{2L}{n}$$ where L is the length of the rod.

Using this equation for **longitudinal vibration** waves, the fundamental harmonic wavelength is λ_1=2L, the second harmonic wavelength is λ_2=L and the third harmonic wavelength is λ_3=2L/3. Now the ratios of the second harmonic mode to the fundamental mode and the third harmonic mode to the fundamental mode can be calculated as follows:

λ_2/λ_1 = L/2L = 0.50 (Ratio of Second Harmonic Mode to Fundamental Harmonic Mode for a **Longitudinal Vibration** wave)

λ_3/λ_1 = (2L/3)/2L = 0.33 (Ratio of Third Harmonic Mode to Fundamental Harmonic Mode for a **Longitudinal Vibration** wave)

FIG. 40 of Sakurai et al. is reproduced below with annotations A, B, $\lambda_1/2$, $\lambda_2/2$ and $\lambda_3/2$.



F I G. 40

FIG. 40 of Sakurai et al. '144 patent shows plots of amplitude of axial displacement of a longitudinal vibration wave versus position for a longitudinal vibration wave traveling along the device. For a given length L, for example between points A and B, along the transmission member, Sakurai et al. follows the equation for the wavelength of longitudinal vibration waves

discussed above. In viewing FIG. 40 of Sakurai et al. between points A and B, λ_1=2L, λ_2=L and λ_3=2L/3. Thus, the ratios for the longitudinal vibration waves $\lambda_2/\lambda_1 = 0.5$ and $\lambda_3/\lambda_1 = 0.33$ hold true for Sakurai et al. which discloses longitudinal vibration modes. Further, FIG. 40 lists the fundamental mode, 2nd harmonic mode and 3rd harmonic mode. The fundamental, 2nd and 3rd modes are harmonically related which means the wavelengths are integer multiples of each other, further evidence that Sakurai et al. discloses longitudinal vibration.

The plot of the amplitude of axial displacement of a longitudinal vibration wave versus position for a longitudinal vibration wave traveling along the device in FIG. 40 of Sakurai et al. '144 patent showing the loops and nodes of the longitudinal vibration motion of Sakurai et al. DOES NOT depict the device physically moving up and down. This plot is an abstract representation of the maximum deviation from equilibrium of a point at the given position along the longitudinal axis of the device. The plot merely shows an amplitude of axial displacement of a longitudinal vibration versus position for a longitudinal vibration traveling along the device to the tip.

In FIGS. 40-42 of the Sakurai et al. '144 patent, the progression of the wavelengths from fundamental mode to 2nd harmonic mode to 3rd harmonic mode shows the integer multiple relationship between the wavelengths described in the preceding analysis, and not the anharmonic relationship between the transverse vibration modes which will be discussed below.

Transverse Vibration Wavelength Equations

The equation for transverse vibration wavelength can be obtained by examining the solution for the mode shapes of a transverse wave:[2]

$$Y = a[\cosh(2\pi\mu\ x) - \cos(2\pi\mu\ x)] + b[\sinh(2\pi\mu\ x) - \sin(2\pi\mu\ x)]$$

[2] Morse, P.M. and Ingard, K.U., *Theoretical Acoustics*, Princeton University Press, Princeton, NJ, pp. 181-182, 1968 (see attached Exhibit A).

where Y is the amplitude of the wave at position x on the rod, a and b are constants depending upon the specific nature of the problem and $2\pi\mu$ is the wavenumber, k, of the transverse vibration wave. Using the relationship between the wavelength and the wavenumber

$$k = 2\pi\mu = \frac{2\pi}{\lambda}$$

gives

$$\lambda = \frac{1}{\mu}$$

Substituting the expression for $\mu_n{}^2$ gives the expression for the wavelength of the n^{th} transverse mode:

$$\lambda_n = \frac{2L}{\beta_n} \qquad \text{with } \beta_n = \begin{cases} 0.597 & n = 1 \\ 1.494 & n = 2 \\ n - \frac{1}{2} & n > 2 \end{cases}$$

where L is the length of the rod.

Using this equation for **transverse vibration waves**, the fundamental mode wavelength is λ_1=3.35L, the second mode wavelength is λ_2=1.34L and the third mode wavelength is λ_3=0.8L. Now the ratios of the second mode to the fundamental mode and the third mode to the fundamental mode can be calculated as follows:

$\boldsymbol{\lambda_2/\lambda_1 = 1.34L/3.35L = 0.40}$ (Ratio of Second Mode to Fundamental Mode for a **Transverse Vibration Wave**)

$\boldsymbol{\lambda_3/\lambda_1 = 0.8L/3.35L = 0.24}$ (Ratio of Third Mode to Fundamental Mode for a **Transverse Vibration Wave**)

Comparing the ratios for longitudinal vibration and transverse vibration waves of the second mode and the third mode to the fundamental mode shows the inequality of the ratios for the varying waves. Therefore, further examination of the inequality of the ratios for the longitudinal

vibration waves and the transverse vibration waves clearly shows that the Sakurai et al. discloses a longitudinal vibration:

(Longitudinal Vibration Waves) (Sakurai et al. '144 Patent)		**(Transverse Vibration Waves)**
$\lambda_2/\lambda_1 = 0.50$	$\neq$	$\lambda_2/\lambda_1 = 0.40$
$\lambda_3/\lambda_1 = 0.33$	$\neq$	$\lambda_3/\lambda_1 = 0.24$

As the above analysis shows, examination of equations for longitudinal vibration waves, the Sakurai et al. '144 specification and FIGS. 40-42 all confirm that Sakurai et al. discloses longitudinal vibration.

<u>Sakurai et al. Discloses Longitudinal Vibration – Stress Equation Confirmation</u>

Further evidencing that Sakurai et al. discloses longitudinal vibration is the specification of the Sakurai et al. '144 patent and well known stress equations. The Sakurai et al. '144 patent specification discusses a statement applicable to longitudinal vibration waves, but <u>not</u> applicable to transverse vibration waves:

> As is evident from FIG. 42, the junction between the horn 3 and the member 4, the first junction 171, and the second junction 177 are located at the loops of the ultrasonic vibration of the vibration-transmitting member 4. As is generally known in the art, less stress is applied to those portion of any member which are located at the loops of ultrasonic vibration, than to those portions which are located at the nodes of the vibration. Hence, a relatively small stress acts on the junction between the horn 3 and the member 4, the first junction 171, and the second junction 177--all being mechanically weak. (Sakurai et al. '144; Col. 25, Line 63 - Col. 26, Line 5).

The above statement is only true for longitudinal vibration waves, and not transverse vibration waves. Therefore, for the longitudinal vibration disclosed in Sakurai et al., there is little stress at the members located at the loops.

The well known equation for stress in a rod deformed by **transverse vibration** is:[3]

$$\sigma = \frac{Mc}{I} = Ec\frac{d^2v}{dx^2}$$

where σ is the stress, M is the bending moment, c is the thickness of the rod, I is the moment of inertia of the rod, E is the Young's modulus of the rod, v is the transverse displacement of the rod, and x is the coordinate along the axis of the rod.

For a rod undergoing transverse vibration in a normal mode, the spatial distribution of the transverse displacement amplitude can be approximated by a sinusoidal wave pattern as follows:

$$v = A\sin(kx)$$

substituting this into the stress equation yields:

$$\sigma = -Eck^2 A\sin(kx)$$

Therefore, for transverse vibration, it is clear that the maximum values for the stress are coincident with the points of maximum displacement ($kx = ((2n+1)\pi)/2$), which are defined as the loops and the minimum values for the stress are coincident with the points of minimum displacement, which are defined as the nodes. Conversely, the longitudinal vibration disclosed in Sakurai et al. incurs maximum stress at the nodes. Therefore, the stress equations confirm the Sakurai et al. '144 patent discloses longitudinal vibration, not transverse vibration.

Sakurai et al. Discloses Longitudinal Vibration And Energy Transfer Occurs At Tip

The Sakurai et al. '144 patent discloses longitudinal vibration that can only efficiently deliver energy to a surrounding medium at its tip. The longitudinal vibration operates by pushing material in an oscillatory fashion along the longitudinal axis of the rod only. This means

[3] Craig, R.R. Jr., *Mechanics of Materials*, John Wiley and Sons, New York, NY, p. 268, 1996 (see attached Exhibit B).

that material along the side of a longitudinally oscillating rod is affected far less than material at the tip. Along the side, the rod is moving parallel to the surface of the adjacent material and little mechanical energy other than frictional heat is transferred from the rod into the surrounding material. At the tip, the rod is moving perpendicular to the surface of the adjacent material and much more mechanical energy can be imparted. Regarding FIG. 40 of the Sakurai et al. '144 patent (reproduced above), the only effective loop for transferring mechanical energy from a longitudinal vibration would be the one at the tip. Any other loop along the device is rendered ineffective because it can only vibrate parallel to the surface of a material with which it is in contact.

Sakurai et al. '144 discloses an ultrasonic treatment apparatus using longitudinal vibration that emulsifies tissue at the tip of the Sakurai et al. device. Sakurai et al. discloses:

> The **tips** of probes of different types, which can be attached to the hand piece, must be vibrated at different amplitudes to emulsify different types of tissues within body cavities, each type with the highest possible efficiency. (Sakurai et al. '144; Col. 12, Lines 24-28) (Emphasis added)
>
> If the **tip** of the probe 61 is in contact with a living tissue within a body cavity, it **emulsifies the tissue**." (Sakurai et al. '144; Col. 13, Lines 16-17)
>
> The **tip** of the member 4 is vibrated, and can therefore **cut or emulsify an affected tissue** or can break stones in a body cavity." (Sakurai et al. '144; Col. 23, Lines 61-63) (Emphasis Added)

Sakurai et al. discloses a device that uses longitudinal vibration. Sakurai et al. does <u>NOT</u> disclose or suggest an **ultrasonic probe supporting a standing transverse wave**. Thus, Applicants respectfully request withdrawal of the obviousness rejections to claims 1-86, and reconsideration and allowance of pending claims 1-22, 24-41, 43-60 and 62-86.

In contrast to Sakurai et al., Applicants' amended independent claims 1, 19, 38, 57 and 78 recite **ultrasonic probe supporting a standing transverse wave**.

Specifically, Applicants' amended independent claim 1 recites:

> 1. A device for treating a **gynecological tissue** adjacent a body lumen comprising:

. . .

wherein the **ultrasonic energy creates a standing transverse wave in the elongated probe such that a plurality of nodes and a plurality of anti-nodes are formed along the longitudinal axis of the elongated probe to treat the tissue** adjacent the body lumen.

Applicants' amended independent 19 recites:

19. A device for treating **gynecological diseases** by removing targeted cells on a surface of a body cavity comprising:

. . .

an ultrasonic generator for providing ultrasonic energy to the member to create **a standing transverse wave in the member forming a plurality of nodes and a plurality of anti-nodes along the longitudinal axis of the member** and generating cavitation along the longitudinal axis of the member to remove targeted cells on the surface of a body cavity.

Applicants' amended independent claim 38 recites:

38. A method of treating a **gynecological tissue** adjacent a body lumen, the method comprising:

. . .

(b) providing ultrasonic energy from an ultrasonic generator to create **a standing transverse wave in the flexible member forming a plurality of nodes and a plurality of anti-nodes to the flexible member** for emission along the longitudinal axis of the flexible member to the tissue adjacent the body lumen.

Applicants' amended independent claim 57 recites:

57. A method for treating **gynecological diseases** by destroying targeted cells on a surface of a body cavity, comprising the steps of:

. . .

(b) providing ultrasonic energy to the member to create **a standing transverse wave in the member forming a plurality of nodes and a plurality of anti-nodes along the longitudinal axis of the member** and generating an area of cavitation along the longitudinal axis of the member;

Applicants' amended independent claim 78 recites:

78. An ultrasonic medical device for treating **gynecological disease** comprising:

...

> wherein the probe supports **a standing transverse wave producing a plurality of nodes and a plurality of anti-nodes along the longitudinal axis to ablate cells** along at least a portion of the longitudinal axis of the probe.

As described above, the Sakurai et al. '144 patent does NOT disclose or suggest an **ultrasonic probe supporting a standing transverse wave** as claimed in Applicants' claimed invention. Thus, Applicants respectfully request withdrawal of the obviousness rejections to claims 1-86, and reconsideration and allowance of pending claims 1-22, 24-41, 43-60 and 62-86.

Double Patenting Rejections

The Office Action rejected claims 1-86 under the judicially created doctrine of obviousness-type double patenting as being unpatentable based on claims of one issued patent and four pending patent applications: U.S. Patent No. 6,524,251; copending Application No. 09/975,725 (now U.S. Patent No. 6,695,782); copending Application No. 09/776,015 (now U.S. Patent No. 6,652,547); copending Application No. 09/625,803 and copending Application No. 09/917,471 (now U.S. Patent No. 6,695,781).

Applicants believe the present application claims separate and distinct subject matter than the four patents and one patent application that form double patenting rejections. Thus, Applicants respectfully request withdrawal of all double patenting rejections because the present application claims subject matter that is patentably distinct from the claimed subject matter in the four patents and one patent application.

The Present Application Claims Subject Matter That Is Patentably Distinct

The MPEP definition of Obviousness-Type Double Patenting states:

> Obviousness-type double patenting requires rejection of an application claim when the claimed subject matter is **not patentably distinct** from the subject matter claimed in a commonly owned patent when the issuance of a second patent would provide unjustified extension of the term of the right to exclude granted by a patent. *See Eli Lilly & Co. v. Barr Labs., Inc.*, 251 F.3d 955, 58 USPQ2d 1865 (Fed. Cir. 2001); *Ex parte Davis*, 56 USPQ2d 1434, 1435-36 (Bd. Pat. App. & Inter. 2000).

MPEP 804(II)(B)(1) (emphasis in original).

Applicants respectfully state that the present application claims subject matter that is patentably distinct from the claimed subject matter in the four patents and one patent application. Thus, the present application is not claiming common subject matter with the four patents and one patent application.

The present application is entitled "Apparatus and Method for Treating Gynecological Diseases Using an Ultrasonic Medical Device Operating in a Transverse Mode." The present application relates to an apparatus and a method for using an ultrasonic medical device operating in a transverse mode to treat gynecological tissue and gynecological disease.

Independent claims 1, 19, 38, 57 and 78 of the present application recite treating gynecological tissue and gynecological disease. The claims in the four patents and one patent application do not contain such a limitation. The limitation is not an obvious variation of the patented claims. Thus, the present application claims subject matter that is patentably distinct from the claimed subject matter in the four patents and one patent application. Thus, Applicants respectfully request withdrawal of all double patenting rejections and reconsideration and allowance of pending claims 1-22, 24-41, 43-60 and 62-86.

Conclusion

With this Amendment, Applicants have made an earnest effort to respond to all issues raised in the Office Action of October 29, 2004, and to place all claims presented in condition for allowance. No Amendment made was for the purpose of narrowing the scope of any claim, unless Applicant has argued herein that such amendment was made to distinguish over a particular reference or combination of references.

Applicants submit that all claims are allowable as written and respectfully request early favorable action by the Examiner. If the Examiner believes that a telephone conversation with Applicant's attorney would expedite prosecution of this application, the Examiner is cordially invited to call the undersigned attorney of record.

Respectfully submitted,

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THEORETICAL ACOUSTICS

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4.3 SIMPLE-HARMONIC OSCILLATIONS

It has been seen in the last section that imposing boundary conditions limits the sorts of motion that a string can have, and that if the boundary conditions correspond to the fixing of both ends of the string to rigid supports, the motion is limited to *periodic* motion. The latter result is an unusual one, for we found in the last chapter that even as simple a system as a pair of coupled oscillators does not, in general, move with periodic motion. It is not unusual for a system to oscillate with simple-harmonic motion (which is a special type of periodic motion) when it is started off properly (we shall see that practically every vibrating system can do this); what is unusual in the string between rigid supports is that *every* motion is periodic, no matter how it is started.

Our problem in this section is to find the possible simple-harmonic oscillations of the string (the normal modes of vibration) and to see what the relation is between the frequencies of these vibrations that makes the resulting combined motion always periodic. The problem of determining the normal modes of vibration of a system is not just an academic exercise. For systems more complicated than that of the string between rigid supports, we have no method of graphical analysis similar to that of the last section, and the only feasible method of discussing the motion is to "take it apart" into its constituent simple-harmonic components. There is also a physiological reason for studying the problem, for the ear itself analyzes a sound into its simple-harmonic parts (if there are any). We distinguish between a note from a violin and a note from a bell, for instance, because of this analysis. If the frequencies present in a sound are all integral multiples of a fundamental frequency, as they are in a violin, the sound seems more musical than when the frequencies are not so simply related, as in the note from a bell.

Traveling and standing waves

We start our discussion with the wave equation (4.1.2), which, we showed determines the motion of a perfectly flexible string as long as it is not displac[ed] too far from equilibrium (as long as $|\partial y/\partial x| \ll 1$).

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad c^2 = \frac{T}{\epsilon} \qquad (4.3[.1])$$

The wave equation corresponds to a number of statements concern[ing] the motion of a string. We saw in the last section that it implies that [a] wave motion travels with its shape unchanged, at a velocity c, independ[ent] of this shape. Since the derivative $\partial^2 y/\partial x^2$ is proportional to the curvatu[re of] the shape of the string at a given instant, Eq. (4.3.1) states that the *acceler[ation]*

f any portion of the string is *directly proportional to the curvature* of that portion. If the curvature is downward, the acceleration is downward, and vice versa; and the greater the curvature, the faster the velocity changes.

If the string is infinite in extent, it can carry waves which travel exclusively n one direction. In that case, as was pointed out at the beginning of this chapter, if the time dependence of the wave is to be sinusoidal, its space dependence must also be sinusoidal. All simple-harmonic waves traveling n the positive x direction must have the form

$$y(x,t) = A\cos\left[\frac{\omega}{c}(x - ct) - \Phi\right]$$

or

$$y(x,t) = C\exp\left[\frac{i\omega}{c}(x - ct)\right] \tag{4.3.2}$$

f $C = Ae^{-i\Phi}$ and if physical meaning is attached only to the real part of the second expression. For a simple-harmonic wave in the negative x direction we substitute $-(x + ct)$ for $(x - ct)$ in these expressions. Incidentally, the reason we have chosen the time factor to be $e^{-i\omega t}$ rather than $e^{j\omega t}$ is that then the sign of the x part of the exponent, $e^{\pm i\omega x/c}$, indicates the direction of the wave.

For the wave of Eq. (4.3.2), the energy and momentum densities are Eqs. (4.1.9) and (4.1.12)]

$$\text{Kinetic energy density} = U = \tfrac{1}{2}\epsilon\omega^2A^2\sin^2\left[\frac{\omega}{c}(x - ct) - \Phi\right]$$

$$\text{Potential energy density} = V = \tfrac{1}{2}T\left(\frac{\omega}{c}\right)^2A^2\sin^2\left[\frac{\omega}{c}(x - ct) - \Phi\right]$$

$$\text{Total energy density} = W_{tt} = \epsilon\omega^2A^2\sin^2\left[\frac{\omega}{c}(x - ct) - \Phi\right] = H \tag{4.3.3}$$

$$\text{Energy flux} = W_{tx} = cH$$

$$\text{Longitudinal momentum density} = W_{xt} = \frac{H}{c}$$

$$\text{Longitudinal stress} = W_{xx} = \frac{H}{c^2}$$

The energy density is greatest where the string's slope and transverse velocity are greatest, each packet of energy spaced a half wavelength from its neighbor, each traveling with a velocity c. Consequently, the energy flux W_{tx} is equal to c times the energy density W_{tt}. The wavelength of these waves is, of

course, the distance between one wave peak and the next, a distance such that an increase of x by λ will increase $(\omega/c)(x - ct)$ by 2π, so that $(\omega/c)\lambda = 2\pi$, or $\lambda = 2\pi c/\omega$.

We could reach the same conclusions by asking what sort of shape the string will have when it vibrates with simple-harmonic motion, i.e., when its time dependence is through the factor $e^{-i\omega t}$. Setting $y(x,t) = Y(x)e^{-i\omega t}$ into the equation of motion (4.1.3) or (4.1.8), we obtain a familiar equation for $Y(x)$.

$$\frac{d^2Y}{dx^2} + \left(\frac{\omega}{c}\right)^2 Y = 0 \qquad c^2 = \frac{T}{\epsilon} \tag{4.3.4}$$

which is identical with Eq. (1.2.1). This is the equation for simple-harmonic dependence on x, with "angular frequency" $k = \omega/c$ and "period" $\lambda = 2\pi/k = 2\pi c/\omega$. The quantity k is called the *wavenumber* of the wave; its dimensions are inverse length. The quantity λ is the wavelength of the wave, the distance from crest to crest of a sinusoidal wave traveling in one direction.

The general solution of Eq. (4.3.4) can be written

$$Y(x) = C_+e^{i\omega x/c} + C_-e^{-i\omega x/c}$$

so that

$$\begin{aligned} y(x,t) &= C_+e^{i\omega(x-ct)/c} + C_-e^{-i\omega(x+ct)/c} \\ &= A_+ \cos\left[\frac{\omega}{c}(x - ct) - \Phi_+\right] + A_- \cos\left[\frac{\omega}{c}(x + ct) + \Phi_-\right] \end{aligned} \tag{4.3.5}$$

representing two waves, of the same frequency and wavelength, traveling in opposite directions along the string. Since the wave equation is linear, neither wave has any effect on the other.

This mutual independence of the waves extends to expressions for their energy-momentum-stress terms, such as the total energy,

$$H = U + V = \omega^2(\theta_+^2 + 2\theta_+\theta_- + \theta_-^2)$$

where

$$\theta_+ = A_+ \sin\left[\frac{\omega}{c}(x - ct) - \Phi_+\right] \qquad \theta_- = A_- \sin\left[\frac{\omega}{c}(x + ct) + \Phi_-\right]$$

The mean value of the square terms is $\langle\theta_+^2\rangle = \frac{1}{2}A_+^2$ and $\langle\theta_-^2\rangle = \frac{1}{2}A_-^2$, neither of which is zero. But the cross terms can be written

$$\theta_+\theta_- = \tfrac{1}{2}A_+A_-\left[\cos(2\omega t + \Phi_+ + \Phi_-) - \cos\left(2\frac{\omega}{c}x - \Phi_+ + \Phi_-\right)\right]$$

When averaged over space and time, the average is zero. The energy flux has no cross term;

$$W_{tx} = -T\frac{\partial y}{\partial t}\frac{\partial y}{\partial x} = \epsilon c\omega^2(\theta_+^2 - \theta_-^2)$$

so that even the instantaneous values of the flux are simply the differences between the two individual fluxes. Thus the average values of the stress-energy tensor are

$$H = \langle W_{tt} \rangle = \tfrac{1}{2}\epsilon\omega^2(A_+^2 + A_-^2) = \frac{1}{c^2}\langle W_{xx} \rangle$$
$$\Upsilon = \langle W_{tx} \rangle = \tfrac{1}{2}\epsilon\omega^2 c(A_+^2 - A_-^2) = \frac{1}{c}\langle W_{xt} \rangle \tag{4.3.6}$$

The energy and stress terms are the sum of the terms arising from each wave. The energy and momentum fluxes are the difference of the terms, since the two waves are flowing in opposite directions.

If the amplitudes of the two simple-harmonic waves are equal, there is no net flow of energy or momentum, and the combination is called a *standing wave*.

$$y(x,t) = C_+ e^{i\omega(x-ct)/c} + C_- e^{-i\omega(x+ct)/c}$$
$$= 2A\cos\left(\frac{\omega}{c}x + \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_-\right)\exp\left(-i\omega t + \tfrac{1}{2}\Phi_+ + \tfrac{1}{2}\Phi_-\right) \quad \text{(real part)}$$
$$= 2A\cos\left(\frac{\omega}{c}x + \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_-\right)\cos\left(\omega t - \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_-\right) \tag{4.3.7}$$

where $C_+ = Ae^{i\Phi_+}$ and $C_- = Ae^{i\Phi_-}$ have the same amplitude but different phases. In this case the shape of the wave does not move along the string; it simply oscillates in amplitude with simple-harmonic motion. At points where $\cos[(\omega/c)x + \frac{1}{2}\Phi_+ - \frac{1}{2}\Phi_-] = 0$, the two traveling waves always cancel each other and the string never moves. These points are called the *nodal points* of the wave motion. In the case that we are considering, where the density and tension are uniform, the nodal points are equally spaced along the string a distance $c/2\nu$ apart, two for each wavelength. Halfway between each pair of nodal points is the part of the string having the largest amplitude of motion, where the two traveling waves always add their effects. This portion of the wave is called a *loop*, or *antinode*.

We should ask how a standing wave gets established and is maintained, for if there is no motion at each node, there can be no flow of energy from one loop to its neighbors. The answer is that a standing wave is a steady-state situation. During the transient state, when energy is being distributed along the string, the nodes are not perfect (that is, y is not exactly zero there) and energy does pass from one loop to the next. Also, even for the steady-state situation, the nodes are only perfect when there is no friction. With zero friction, once a loop has acquired its energy, it can oscillate forever. If friction is present, the "nodes" are simply places of minimal (but not zero) amplitude of vibration; some energy flows from loop to loop.

Normal modes

So far, we have neglected boundary conditions. If we require that $y = 0$ when $x = 0$, the general form of (4.3.5) can no longer be used; the number of possible harmonic motions is limited. The expression for y that must be used is the standing-wave form (4.3.7) with the angles Φ so chosen that *a nodal point coincides with the point of support* $x = 0$:

$$y = A \sin\left(\frac{2\pi\nu}{c} x\right) \cos\left(2\pi\nu t - \Phi\right) \tag{4.3.8}$$

This agrees with the discussion in the previous section. For the simple boundary condition that we have used, the reflected wave has the same amplitude as the incident wave; and when the incident one is sinusoidal, the result is a set of standing waves. Any frequency is allowed, however.

When the second boundary condition $y = 0$ at $x = l$ is added, the number of possible simple-harmonic motions is still more severely limited. For now, of all the possible standing waves indicated in (4.3.8), *only those which have a nodal point at* $x = l$ *can be used.* Since the distance between nodal points depends on the frequency, the string fixed at both ends cannot vibrate with simple-harmonic motion of any frequency; only a discrete set of frequencies is allowed, the set that makes $\sin [(2\pi\nu/c)l]$ zero. The distance between nodal points must be l, or it must be $l/2$ or $l/3$, etc. The allowed frequencies are therefore $c/2l$, $2c/2l$, $3c/2l$, etc., and the different allowed simple-harmonic motions are all given by the expression

$$\begin{aligned} y &= A_n \sin\left(\frac{\pi n x}{l}\right) \cos\left(\frac{\pi n c}{l} t - \Phi_n\right) \qquad n = 1, 2, 3, 4, \ldots \\ \nu_n &= \frac{nc}{2l} = \frac{n}{2l}\sqrt{\frac{T}{\epsilon}} \end{aligned} \tag{4.3.9}$$

The lowest allowed frequency $\nu_1 = c/2l$ is called the *fundamental frequency* of vibration of the string. It is the frequency of the general periodic motion of the string, as we showed in the last section. The higher frequencies are called *overtones*, the first overtone being ν_2, the second ν_3, and so on.

The equation for the allowed frequencies given in Eq. (4.3.9) expresses an extremely important property of the uniform flexible string stretched between rigid supports. It states that the frequencies of all the overtones of such a string are *integral multiples of the fundamental frequency.* Overtones bearing this simple relation to the fundamental are called *harmonics*, the fundamental frequency being called the first harmonic, the first overtone (twice the fundamental) being the second harmonic, and so on.

Very few vibrating systems have harmonic overtones, but these few are the bases of nearly all musical instruments. For when the overtones are harmonic, the sound seems particularly satisfying, or musical, to the ear.

The general solution of this is

$$y = C_1 e^{2\pi\mu x} + C_2 e^{-2\pi\mu x} + C_3 e^{2\pi i\mu x} + C_4 e^{-2\pi i\mu x}$$
$$= a \cosh(2\pi\mu x) + b \sinh(2\pi\mu x) + c \cos(2\pi\mu x) + d \sin(2\pi\mu x) \quad (5.1.11)$$

where $\cosh u = \cos(iu)$ and $\sinh u = -i \sin(iu)$. See Eq. (1.2.10) and Tables I and II.

This general solution satisfies Eq. (5.1.10) for any value of the frequency ν. It is, of course, the boundary conditions that pick out the set of allowed frequencies.

Bar clamped at one end

For example, if we have a bar of length l clamped at one end $x = 0$, the boundary conditions at this end are that *both* y and its slope $\partial y/\partial x$ must be zero at $x = 0$. The particular combination of the general solution (5.1.11) that satisfies these two conditions is the one with $c = -a$ and $d = -b$.

$$Y = a[\cosh(2\pi\mu x) - \cos(2\pi\mu x)] + b[\sinh(2\pi\mu x) - \sin(2\pi\mu x)] \quad (5.1.12)$$

If the other end is free, y and its slope will not be zero, but the bending moment $M = QS\kappa^2(d^2Y/dx^2)$ and the shearing force $F = -QS\kappa^2(d^3Y/dx^3)$ must both be zero, since there is no bar beyond $x = l$ to cause a moment or a shearing stress. We see that *two* conditions must be specified for each end instead of just one, as in the string. This is due to the fact that the equation for Y is a fourth-order differential equation, and its solution involves four arbitrary constants whose relations must be fixed, instead of two for the string. It corresponds to the physical fact that whereas the only internal stress in the string is tension, the bar has two, bending moment and shearing force, each depending in a different way on the deformation of the bar.

The two boundary conditions at $x = l$ can be rewritten as $\dfrac{1}{4\pi^2\mu^2}\dfrac{d^2Y}{dx^2} = 0$ and $\dfrac{1}{8\pi^3\mu^3}\dfrac{d^3Y}{dx^3} = 0$ at $x = l$. Substituting expression (5.1.12) in these, we obtain two equations that fix the relationship between a and b and between μ and l:

$$a[\cosh(2\pi\mu l) + \cos(2\pi\mu l)] + b[\sinh(2\pi\mu l) + \sin(2\pi\mu l)] = 0$$
$$a[\sinh(2\pi\mu l) - \sin(2\pi\mu l)] + b[\cosh(2\pi\mu l) + \cos(2\pi\mu l)] = 0$$

or

$$b = a\frac{\sin(2\pi\mu l) - \sinh(2\pi\mu l)}{\cos(2\pi\mu l) + \cosh(2\pi\mu l)} = -a\frac{\cos(2\pi\mu l) + \cosh(2\pi\mu l)}{\sin(2\pi\mu l) + \sinh(2\pi\mu l)} \quad (5.1.13)$$

By dividing out a and multiplying across, we obtain an equation for μ:

$$[\cosh(2\pi\mu l) + \cos(2\pi\mu l)]^2 = \sinh^2(2\pi\mu l) - \sin^2(2\pi\mu l)$$

Utilizing some trigonometric relationships, this last equation can be reduced to two simpler forms:

$$\cosh(2\pi\mu l)\cos(2\pi\mu l) = -1 \qquad \text{or} \qquad \coth^2(\pi\mu l) = \tan^2(\pi\mu l) \tag{5.1.14}$$

where $\coth z = \cosh z/\sinh z$.

The allowed frequencies

We shall label the solutions of this equation in order of increasing value. They are $2\pi\mu_1 l = 1.8751$, $2\pi\mu_2 l = 4.6941$, $2\pi\mu_3 l = 7.8548$, etc. To simplify the notation, we let $1/\pi$ times the numbers given above have the labels β_n, so that

$$\mu_n = \frac{\beta_n}{2l} \tag{5.1.15}$$

where $\beta_1 = 0.597$, $\beta_2 = 1.494$, $\beta_3 = 2.500$, etc. It turns out that β_n is practically equal to $n - \frac{1}{2}$ when n is larger than 2.

By fixing μ, we fix the allowed values of the frequency. Using Eq. (5.1.10), we have

$$\nu_n = \frac{\gamma^2\mu_n^2}{2\pi} = \frac{\pi}{2l^2}\sqrt{\frac{Q\kappa^2}{\rho}}\,\beta_n^2 \tag{5.1.16}$$

or

$$\nu_1 = \frac{0.55966}{l^2}\sqrt{\frac{Q\kappa^2}{\rho}} \qquad \begin{aligned} \nu_2 &= 6.267\nu_1 \\ \nu_3 &= 17.548\nu_1 \\ \nu_4 &= 34.387\nu_1 \\ &\cdots\cdots \end{aligned}$$

Notice that the allowed frequencies depend on the inverse *square* of the length of the bar, whereas the allowed frequencies of the string depend on the inverse first power.

Equation (5.1.16) shows how far from harmonics are the overtones for a vibrating bar. The first overtone has a higher frequency than the sixth harmonic of a string of equal fundamental. If the bar were struck so that its motion contained a number of overtones with appreciable amplitude, it would give out a shrill and nonmusical sound. But since these high-frequency overtones are damped out rapidly, the harsh initial sound will quickly change to a pure tone, almost entirely due to the fundamental. A tuning fork can be considered to be two vibrating bars, both clamped at their lower ends. The fork exhibits the preceding behavior, the initial metallic "ping" rapidly dying out and leaving an almost pure tone.

The characteristic functions

The characteristic function corresponding to the allowed frequency ν_n is given by the equation

$$\psi_n = a_n\left(\cosh\frac{\pi\beta_n x}{l} - \cos\frac{\pi\beta_n x}{l}\right) + b_n\left(\sinh\frac{\pi\beta_n x}{l} - \sin\frac{\pi\beta_n x}{l}\right) \tag{5.1.17}$$

where

$$-b_n = a_n \frac{\cosh(\pi\beta_n) + \cos(\pi\beta_n)}{\sinh(\pi\beta_n) + \sin(\pi\beta_n)} = a_n \frac{\sinh(\pi\beta_n) - \sin(\pi\beta_n)}{\cosh(\pi\beta_n) + \cos(\pi\beta_n)}$$

We shall choose the value of a_n so that $\int_0^l \psi_n^2\,dx = l/2$, by analogy with the sine functions for the string. The resulting values for a_n and b_n are $a_1 = 0.707$, $b_1 = -0.518$, $a_2 = 0.707$, $b_2 = -0.721$, $a_3 = 0.707$, $b_3 = -0.707$, etc. For n larger than 2, both a_n and b_n are practically equal to $1/\sqrt{2}$. Some of the properties of these functions that will be of use are

$$\int_0^l \psi_m(x)\psi_n(x)\,dx = \begin{cases} 0 & m \neq n \\ \dfrac{l}{2} & m = n \end{cases} \qquad \psi_n(l) = (-1)^{n-1}\sqrt{2}$$

$$\left(\frac{d\psi_1}{dx}\right)_{x=l} = 1.040\,\frac{\pi\beta_1}{l} \qquad \left(\frac{d\psi_2}{dx}\right)_{x=l} = -1.440\,\frac{\pi\beta_2}{l} \tag{5.1.18}$$

$$\left(\frac{d\psi_n}{dx}\right)_{x=l} \simeq (-1)^{n-1}\sqrt{2}\,\frac{\pi\beta_n}{l} \qquad \text{and} \qquad \beta_n \simeq n - \tfrac{1}{2} \qquad n > 2$$

$$\psi_n \simeq \frac{1}{\sqrt{2}}\left[e^{-\pi\beta_n x/l} + (-1)^{n-1}e^{\pi\beta_n(x-l)/l}\right] + \sin\left(\frac{\pi\beta_n x}{l} - \frac{\pi}{4}\right) \qquad n > 2$$

The shapes of the first five characteristic functions are shown in Fig. 5.4. Note that for the higher overtones most of the length of the bar has the sinusoidal shape of the corresponding normal mode of the string, with the nodes displaced toward the free end. In terms of the approximate form given above for ψ_n, the sine function is symmetrical about the center of the bar; the first exponential alters the sinusoidal shape near $x = 0$ enough to make ψ_n have zero value and slope at this point; and the second exponential adds enough near $x = l$ to make the second and third derivatives vanish. Note also that the number of nodal points in ψ_n is equal to $n - 1$, as it is for the string.

In accordance with the earlier discussion of series of characteristic functions, we can now show that a bar started with the initial conditions, at $t = 0$, of $y = y_0(x)$ and $\partial y/\partial t = v_0(x)$ will have a subsequent shape given by the series

$$y = \sum_{n=1}^{\infty} \psi_n(x)[B_n \cos(2\pi\nu_n t) + C_n \sin(2\pi\nu_n t)] \tag{5.1.19}$$

where

$$B_n = \frac{2}{l}\int_0^l y_0(x)\psi_n(x)\,dx$$

$$C_n = \frac{1}{\pi\nu_n l}\int_0^l v_0(x)\psi_n(x)\,dx$$

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6.3 FLEXURAL STRESS IN LINEARLY ELASTIC BEAMS

In the previous section, assumptions were made about the geometry of deformation of slender beams, and an expression for the resulting extensional strain ϵ_x was derived, Eq. 6.3. The corresponding normal stress in beams, σ_x, is often called the *flexural stress*. To obtain an expression for the flexural stress, we need to consider the material behavior, that is, the stress-strain-temperature behavior of the material. To simplify our initial study of stresses in beams, let us assume that the material is linearly elastic and isotropic, and that the temperature remains constant. Then, the following two assumptions permit us to determine the flexural stress σ_x:

1. The material obeys Hooke's law, Eq. 2.32a, with $\Delta T = 0$.
2. The transverse normal stresses, σ_y and σ_z, may be neglected in comparison with the primary normal stress, σ_x.

By combining these two assumptions, we find that the uniaxial stress-strain equation

$$\sigma_x = E\epsilon_x \tag{6.6}$$

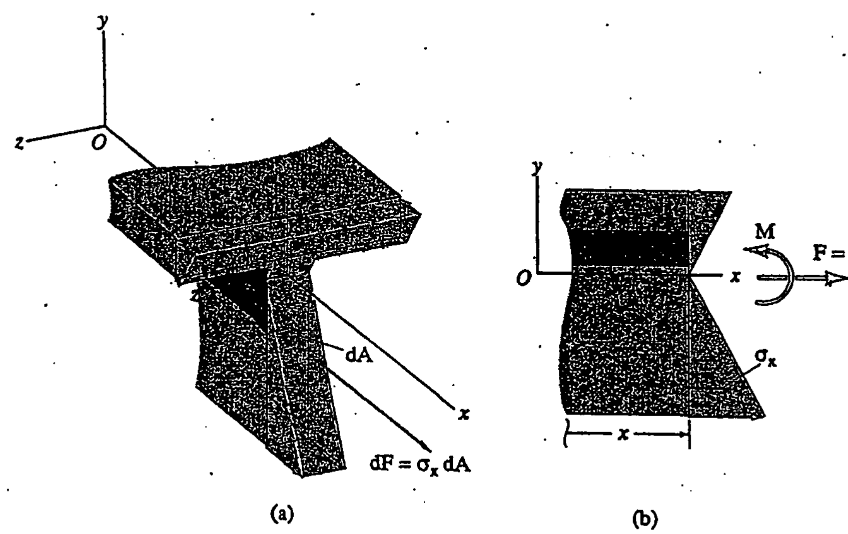
applies to bending of linearly elastic beams. When Eqs. 6.3 and 6.6 are combined, we obtain the following expression

$$\sigma_x = \frac{-Ey}{\rho} \tag{6.7}$$

If E = const, or if $E = E(x)$, the normal stress on a cross section is linear in y, as given by Eq. 6.8 and indicated in Fig. 6.10.[3]

[redacted] (6.8)

GURE 6.10 The flexural :ess distribution at a cross ction where $\rho(x)$ is positive.



[3]In Section 6.5 we will consider stresses in nonhomogeneous beams, that is, stresses in beams that are made of more than one material.

As indicated in Fig. 6.10, the stress resultants that are related to the normal stress σ_x acting on the cross section are:

$$F(x) = \int_A \sigma_x \, dA, \qquad M(x) = -\int_A y\sigma_x \, dA \tag{6.9}$$

A positive moment produces compression in the $+y$ fibers of the beam.

In Section 9.4 we will consider axial deformation combined with bending, but, for the present discussion of bending alone, let $F \equiv 0$. Therefore, substituting Eq. 6.8 into Eqs. 6.9, we get

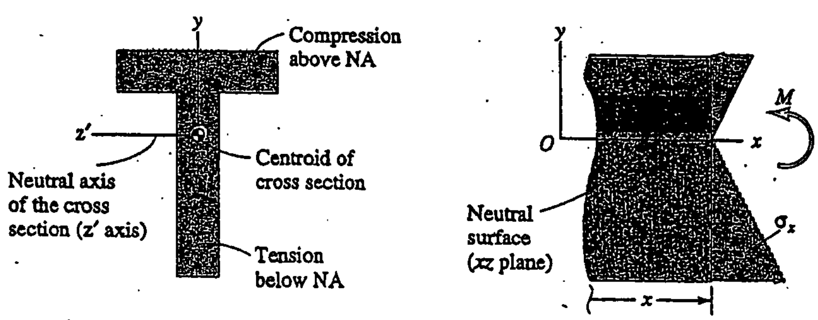
$$F = -\frac{E}{\rho}\int_A y \, dA = 0, \qquad M = \frac{E}{\rho}\int_A y^2 \, dA \tag{6.10}$$

The integrals appearing in Eqs. 6.10 are section properties that are defined in Appendix C:

$$\int_A dA = A, \quad \int_A y \, dA = \bar{y}A, \quad \int_A y^2 \, dA = I_z \tag{6.11}$$

where A is the cross-sectional area, $\bar{y}$ is the y coordinate of the *centroid* of the cross section, and I_z is the *area moment of inertia* about the z axis of the cross section.

In order to satisfy the condition $F = 0$, we must make $\bar{y} = 0$. That is, the z axis of the cross section (labeled the z' axis in Figs. 6.10*a* and 6.11*a*) must pass through the centroid of the cross section. Thus, **the x axis passes through the centroid of each cross section of the undeformed beam.** The z' axis is called the *neutral axis of the cross section,* or simply the *neutral axis (NA)*, because it is the boundary between the portion of the cross section that is in compression and the portion that is in tension, as indicated in Fig. 6.11.[4]



(*a*) Cross section. (*b*) Profile view.

FIGURE 6.11 (*a*) The location of the neutral axis of the cross section, and (*b*) the flexural stress distribution for a homogeneous beam in bending.

[4] In the future, the "z axis of the cross section" will just be labeled z, not z', even though the true z axis does not lie in the particular cross section under consideration.

Combining Eqs. 6.10b and 6.11c, we obtain the *moment-curvature equation* of Bernoulli-Euler beam theory, namely

[redacted] **Moment-curvature equation** (6.12)

The *curvature* $\kappa(x)$ is related to the *radius of curvature* $\rho(x)$ by $\kappa(x) = \dfrac{1}{\rho(x)}$. The product EI is called the *flexural rigidity* of the beam. (In Eq. 6.12 the subscript has been dropped from I_z to simplify the remainder of the discussion of bending of symmetric beams. Subscripts will be needed again in the Section 6.6 on Unsymmetric Bending.)

We can relate the moment-curvature equation, Eq. 6.12, to the deformed-beam segments in Fig. 6.8 by noting that a positive bending moment, $M(x)$, leads to a positive value of $\rho(x)$, which means that the beam is concave upward, as shown in Fig. 6.8*a*. Conversely, a negative moment produces a negative curvature, which means that the center of curvature lies in the $-y$ direction, as shown in Fig. 6.8*b*.

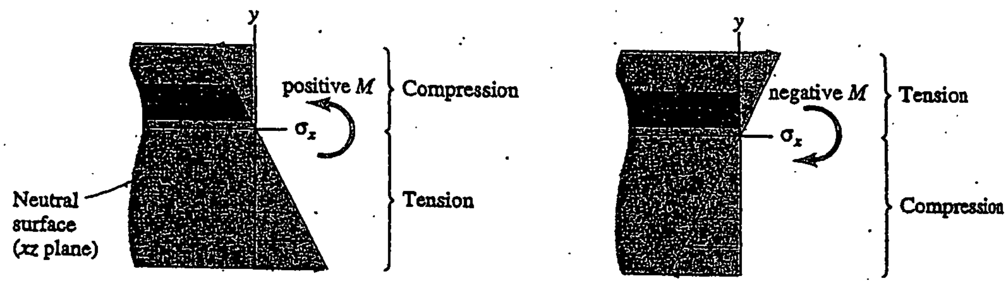
Finally, Eqs. 6.8 and 6.12 may be combined to give the important *flexure formula* of Bernoulli-Euler beam theory.[5]

[redacted] **Flexure formula** (6.13)

By making the assumptions that plane sections remain plane and that the material is linearly elastic with $E = E(x)$, we have obtained an expression for the stress distribution on a cross section subjected to bending moment $M(x)$. This is the linear stress distribution illustrated in Fig. 6.12.[6]

An assumption made in the derivation of the flexure formula, Eq. 6.13, is that σ_x is much greater than either σ_y or σ_z. It is left as an exercise for the reader to show that this is a reasonable assumption if the beam is long in comparison with its cross-sectional dimensions. (Homework Problem 6.3-36)

FIGURE 6.12 The flexural stress distribution in a linearly elastic beam.



(*a*) Positive moment.

(*b*) Negative moment.

[5] According to the sign convention adopted in this text and illustrated in Fig. 5.4, a positive moment produces compression in the $+y$ fibers of the beam. This results in a minus sign in Eq. 6.13. Some textbooks adopt a different sign convention that leads to a plus sign in the flexure formula.

[6] Compressive stresses as well as tensile stresses may be shown acting on the cross section, as in Fig. 6.11*b*. However, to emphasize here that σ_x is linear in y, compressive stresses are shown in Fig. 6.12 as a continuation of the straight-line plot that depicts tensile stresses.

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